WORKING PAPER

Endogenous Redistributive Cycles — An Overlapping Generations Approach to Social Conflict and Cyclical Growth

by

Christiane Clemens and Maik Heinemann

University of Lüneburg
Working Paper Series in Economics

No. 5

March, 2005

www.uni-lueneburg.de/vwl/papers
ISSN 1860–5508

Endogenous Redistributive Cycles

An Overlapping Generations Approach to Social Conflict and Cyclical Growth

Christiane Clemens* Maik Heinemann**

March 17, 2005

Abstract

This paper discusses the emergence of endogenous redistributive cycles in a stochastic growth model with incomplete asset markets and heterogeneous agents, where agents vote on the degree of progressivity in the tax–transfer–scheme. The model draws from Bénabou (1996) and ties the bias in the distribution of political power to the degree of inequality in the society, thereby triggering redistributive cycles which then give rise to a nonlinear, cyclical pattern of savings rates, growth and inequality over time.

Keywords: Inequality, growth, political cycles, redistribution, Hopf bifurcation
JEL–classification: D31, E62, O41, P16

*University of Hannover, Economics Department, D–30 167 Hannover, Germany, e–mail: *clemens@vwl.uni-hannover.de*.

**University of Lüneburg, Institute of Economics, D–21 332 Lüneburg, Germany, e–mail: *maik.heinemann@uni-lueneburg.de*.

1 Introduction

In this paper, we present a model of stochastic growth with incomplete asset markets and heterogeneous agents, where redistributive cycles and cyclical growth emerge as an outcome of voting processes over public tax–transfer schemes. The focus on distributional conflicts in an endogenous growth setting relates our work to recent research in the field of politico–economic analysis and income distribution.[1]

Regarding the political decision rule, a substantial part of the literature treats this as exogenously given. The commonly employed approach is the one of simple majority voting, where the median voter is the pivotal individual and his preferences determine the outcome of the democratic process; see for instance Bertola (1993), Saint-Paul and Verdier (1993, 1997), Persson and Tabellini (1994), Alesina and Rodrik (1994), Piketty (1995), Krusell *et al.* (1997), or more recently Plümper and Martin (2003). However, as Bourguignon and Verdier (2000) point out, the political economy decision mechanism may itself be endogenous to the process of economic development.

By now, only a limited number of contributions extend the analysis in this direction. Bourguignon and Verdier (2000) discuss an endogenously determined process of democratization in an oligarchy–ruled economy. Ades and Verdier (1996) consider entry costs to the decision making elite, while Gradstein and Justman (1995) assume that the franchise is limited by the level of income. Political participation then increases throughout the growth process. Acemoglu and Robinson (2000) discuss a model, where the rich may choose to extend the franchise in order to prevent the economy from insurrection and associated threats of expropriation.

In this article, we discuss this issue in a model of cyclical dynamics which occur because of the dynamic interaction between political and economic decisions. We consider it a worthwhile extension to make the political decision mechanism endogenous to the dynamics of inequality along the growth path. Our analysis is related to the framework developed by Bénabou (1996, 2000), which displays the convenient feature that the behavioral relationships between the macroeconomic variables are grounded in the intertemporal optimization decisions of single agents, while preserving analytical tractability and allowing for closed–form solutions of the income dynamics and the endogenously determined wealth distribution.

[1] For surveys see Aghion *et al.* (1999) or Drazen (2000, ch. 11).

We follow Bénabou (1996, 2000) in assuming that the political influence is unevenly distributed in the society and pressure groups have the power to enforce redistributive policies which are favorable to them. Yet, contrary to Benabou's approach, we do not fix the rank of the critical pressure group in the wealth distribution at an exogenous *ad hoc* level, but allow for endogenous shifts in the political bias by tying it to the dynamics of income inequality.

Altogether, this indicates a dynamic process, where the distribution of political power is shifting over time. In order to generate redistributive cycles it is sufficient to assume that the bias in public decision–making is moving towards the poor, if the society is highly polarized and wealth inequality is large, while political power shifts towards the upper income classes, if redistribution becomes too equalizing.

Our argument is motivated by various competing forces affecting the agents' voting behavior: On the one hand, redistribution provides an insurance against unfavorable outcomes and therefore is preferred by risk averse but, moreover, also by inequality averse agents, who value comparably egalitarian societies with a low degree of income mobility. Stronger social affinity then goes along with a larger amount of redistribution (Lindert, 1996). On the other hand, to the extent past incomes determine the current level of income and random income components are diversified, individual income mobility is limited. Consequently, an agent facing relatively small prospects of upward income mobility might tend to vote against redistribution (Bénabou and Ok, 2001). This effect can be reinforced if individuals also care for their social status (Corneo and Grüner, 2000).

The link between wealth inequality and the bias in political participation is established exogenously, but can be motivated in several ways: First of all, one might argue that the rich have advantages in building up pressure groups, for instance, by employing networks, whereas the poor are less organized. Furthermore, it is easier for the rich to raise funds for lobbying activities. Empirical evidence suggests a comparably small degree of political participation in the lower income classes; see Shields and Goidel (1997) or Bénabou (2000, and references therein). The low polling rates of the poor can be explained with the presence of opportunity costs, i. e. the poor are primarily concerned with earning their living, as well as with the presence of a certain apathy or frustration regarding the political process. Apart from this, the motivation to engage actively in the political process might be less pronounced in a relatively egalitarian society. Contrary, if the perceived extent of inequality becomes too large, inequality aversion might cause an increase in political participation of the lower income classes, where the threat of a socio–politically instable economy even lets the rich support a certain degree of redistribution (Benhabib and Rustichini, 1996).

Individual income mobility in our model stems from the realization of idiosyncratic shocks in the presence of credit constraints and imperfect markets for pooling risks. From the literature it is well–known that, in this case, redistribution may enhance growth (cf. Banerjee and Newman, 1991; Aghion and Bolton, 1992; Galor and Zeira, 1993; Perotti, 1993; Piketty, 1997; Krusell and Smith, 1998; Matsuyama, 2000).

Redistributive politics then affect growth via associated adjustments in the individual savings rates. Combined with a voting system, where the future distribution of political power is endogenously determined by the current state of wealth inequality, redistributive cycles trigger growth cycles, such that periods of high growth and a low degree of redistribution take turns with intervals of heavy redistribution and correspondingly low growth. In this context, the nonlinear patterns of savings and growth rates observed in our model stand in the tradition of the contributions of Kaldor (1940) and Goodwin (1951), although it is important to stress that, here, cycles stem from voting processes over redistributive tax schemes instead of arising from imbalances between saving and investment or class conflicts in the classical sense.

The paper is organized as follows: Section 2 develops the general framework where income and wealth dynamics are derived within an overlapping generations setting. Section 3 introduces the political mechanism of biased majority voting, where we assume the political weight of a single agent to depend on his absolute level of wealth. We start this section with providing results related to existence and stability of equilibria for the case of an exogenously fixed political bias before turning to the actual point of interest, namely the endogenously determined political mechanism, which ultimately gives rise to redistributive cycles. Section 4 concludes. For easier readability, all proofs are relegated to the Appendix.

2 Income dynamics and redistribution in an OG growth model

Our analysis contributes to the strand of research explaining distributional dynamics as the outcome of stochastic processes in a dynastic context; see

e. g. Becker and Tomes (1979) or Loury (1981). The underlying framework draws from Bénabou (1996, 2000). Utility maximizing individuals choose their preferred extent of redistribution and vote on tax–transfer–schemes in a collective decision process. Growth and redistributive cycles occur, if we introduce deviations from simple majority voting by assuming a biased political process, and, in particular, if we allow this bias to vary over time. Depending on the outcome of the voting process, the amount to be redistributed varies continuously. The model also shows the well–known tradeoff between equity and efficiency (Mirrlees, 1971). Redistribution comes at the cost of a lower endogenously determined growth rate.

The model The economy is populated by a continuum of overlapping generations families, indexed by $i \in [0,1]$. We disregard population growth, i. e. each agent has exactly one offspring. Agents have preferences defined over their own consumption c_t^i, as well as their child's income y_{t+1}^i

$$U_t^i = \ln c_t^i + \gamma \ln y_{t+1}^i \, . \tag{1}$$

$\gamma > 0$ denotes the utility weight, the parents attach to their children's future income endowment. Additionally, we consider an incomplete market economy, where credit markets are missing. The income of agent i depends on her stock of (human or physical) capital k_t^i, the average stock of capital κ_t, as well as an idiosyncratic random component u_t^i:

$$y_t^i = u_t^i A \, (k_t^i)^\beta \kappa_t^{1-\beta} , \qquad \ln u_t^i \sim \mathcal{N}(-\sigma_u^2/2, \sigma_u^2) \, . \tag{2}$$

Here, $A > 0$ denotes the usual productivity parameter. The production technology (2) is concave in the individual variables. In the spirit of Romer (1986), the average stock of capital κ_t represents the level of technical knowledge available in the economy and is enhanced by individual capital investments. Altogether, the technology (2) meets the conditions for ongoing growth of per capita incomes.

Parents are able to invest in the capital stock of their children. However, a redistributive system maps the agent's savings x_t^i into the child's capital endowment according to the following scheme:

$$\hat{x}_t^i = {x_t^i}^{1-\tau_t} \tilde{x}_t^{\tau_t} , \tag{3}$$

borrowed from Bénabou (1996, 2000), and previously employed for instance by Feldstein (1969) and Kanbur (1979). Here, $\hat{x}_t^i$ denotes post–tax investment. The progressivity of this system is measured by the elasticity of post–tax investment τ_t. For $\tau > 0$, the marginal rate rises with pretax investment, for $\tau < 0$, the scheme is regressive. The break–even level, $\tilde{x}_t$, separates the winners from the tax–transfer–system from the losers, by defining the margin, where pre– and post–tax investment are equal and the associated household receives a zero net gain from redistribution. $\tilde{x}_t$ is determined by the government's budget constraint which requires net transfers summing to zero:

$$\int_0^1 x_t^i \, \mathrm{d}i = \int_0^1 {x_t^i}^{1-\tau_t} \tilde{x}_t^{\tau_t} \, \mathrm{d}i = \int_0^1 \hat{x}_t^i \, \mathrm{d}i \, . \tag{4}$$

The offspring's capital stock k_{t+1}^i is also subject to an individual 'ability' shock z_t^i, which is lognormally distributed:

$$k_{t+1}^i = z_t^i \hat{x}_t^i \, , \qquad \ln z_t^i \sim \mathcal{N}(-\sigma_z^2/2, \sigma_z^2) \, . \tag{5}$$

The major difference between the two shocks, u_t^i, z_t^i, is their date of realization. The underlying timing of shocks allows us to explicitly take account of the insurance property of redistribution (Varian, 1980). While u_t^i occurs before redistributive measures are effective, the second shock, z_t^i, takes place afterwards, by this remaining an uninsurable idiosyncratic risk.

Utility maximization of agent i with respect to c_t^i and x_t^i, subject to the budget constraint $y_t^i = c_t^i + x_t^i$, the production function (2), and the redistributive scheme (3), then implies that the all agents save the identical fraction $s(\tau_t)$ of their income:

$$x_t^i = s(\tau_t) y_t^i = \frac{\beta\gamma(1-\tau_t)}{1+\beta\gamma(1-\tau_t)} y_t^i \, . \tag{6}$$

The savings rate, $s(\tau_t)$, depends on the parameter measuring tax progression τ_t, thereby reflecting the well–known result that individual decisions are distorted by the presence of a redistributive tax system, with $s'(\tau) < 0$. Since we argue within an endogenous growth framework, this distortion consequently reduces the long–run growth rate of the economy, thus causing the above mentioned efficiency costs of redistribution (cf. Mirrlees, 1971).

Under the assumption of lognormally distributed exogenous disturbances, the resulting wealth distribution is lognormal, whenever the initial distribution is lognormal too, as we have already mentioned above. We assume $\ln k_t^i \sim \mathcal{N}(\mu_t, \sigma_t^2)$, where σ_t^2 denotes the variance of $\ln k_t^i$ in period t, measuring wealth inequality. Average wealth is then given by $\ln \kappa_t = \mu_t + \frac{\sigma_t^2}{2}$, where μ_t denotes mean (log) wealth. By using equations (2), (5), and (6), we obtain a stochastic difference equation, describing the evolution of (log) wealth over time for family i (see the Appendix for derivation):

$$\ln k_{t+1}^i = \ln z_t^i + (1-\tau_t)\ln u_t^i + \ln s(\tau_t) + \ln A + (1-\tau_t)\beta \ln k_t^i + [1-\beta(1-\tau_t)]\mu_t + \left[\beta^2\tau_t(2-\tau_t)+1-\beta\right]\frac{\sigma_t^2}{2} + \tau(1-\tau_t)\frac{\sigma_u^2}{2}\,. \tag{7}$$

Equation (7) completely describes the dynamics of the wealth distribution for a given redistributive scheme. In each period, wealth is lognormally distributed, that is $\ln k_t^i \sim \mathcal{N}(\mu_t, \sigma_t^2)$ with mean μ_t and wealth inequality σ_t^2 evolving according to:

$$\mu_{t+1} = \mu_t + \ln s(\tau_t) + \ln A - \frac{1}{2}\left[(1-\tau_t)^2\sigma_u^2 + \sigma_z^2\right] + \left[\beta^2\left(1-(1-\tau_t)^2\right)+1-\beta\right]\frac{\sigma_t^2}{2}\,, \tag{8a}$$

$$\sigma_{t+1}^2 = \sigma_z^2 + (1-\tau_t)^2\left[\sigma_u^2 + \beta^2\sigma_t^2\right]\,. \tag{8b}$$

Mean wealth dynamics in general are negatively related to risk. The impact of the parents' risk (σ_u^2) is mitigated by the redistributive system, thereby reflecting the insurance property of taxation, whereas the offspring's risk (σ_z^2) cannot be diversified. As usual, mean wealth increases with a rise in the propensity to save.

If we look at the evolution of wealth inequality (8b), it becomes obvious how an increase in the progressivity of the tax system reduces wealth inequality.[2] While the effects from the initial wealth inequality and from the individual production risk of the parent generation on the resulting wealth distribution are weakened, the effect of the ability shocks affecting the future generation is left unchanged. This outcome can be ascribed to the fact that the underlying redistributive system does not provide an insurance against these shocks.

The growth rate of income Since we are dealing with a typical model of endogenous growth, the growth rate of average income depends on several factors, the first being the endogenously determined propensity to save, which, indirectly, also establishes a link between the degree of tax progression and growth. Because we assumed imperfect capital markets, differences in the marginal productivity of the individual capital stocks are not leveled out by borrowing and lending. For this reason, the growth rate is also affected by the distribution of wealth. The assumed concavity of the

[2] Stability of (8b) requires the set of feasible tax rates which are consistent with a political equilibrium to be bounded below, that is, we restrict our analysis to $\tau \in (\underline{\tau}, 1]$, with $\underline{\tau} = 1 - 1/\beta$.

production function, i. e. decreasing returns with respect to individual inputs, implies that a more unequal distribution of wealth goes along with smaller average output; see Aghion *et al.* (1999, p. 1624) and Bénabou (2000). These two effects appear in the following definition of the growth rate of average income $g_{y,t+1} \equiv \Delta \ln \overline{y}_{t+1}$:

$$g_{y,t+1} = \ln A + \ln s(\tau_t) - \tfrac{1}{2}\beta\,(1-\beta)\,\sigma_z^2 + -\tfrac{1}{2}\beta(1-\beta)\,(1-\tau_t)^2\,\left(\sigma_u^2 + \beta^2\,\sigma_t^2\right) \quad (9)$$

Proposition 1 (Growth effects of inequality and taxation)

(i) The growth rate of average income unambiguously rises with a decrease in wealth inequality, $\partial g_{y,t+1}/\partial \sigma_t^2 < 0$.

(ii) The response of the growth rate of average income to a change in the degree of tax progression is of ambiguous sign. With $s'(\tau_t) < 0$*,*

$$\frac{\partial g_{y,t+1}}{\partial \tau_t} \gtreqless 0 \quad \Longleftrightarrow \quad -\frac{s'(\tau_t)}{s(\tau_t)} \lesseqgtr \beta(1-\beta)(1-\tau_t)\left(\sigma_u^2 + \beta^2 \sigma_t^2\right) \ .$$

The growth rate increases if the marginal (positive) efficiency effect from a more equal distribution of wealth outweighs the marginal (negative) incentive effect of taxation on savings and vice versa.

As can be seen from equation (9), a more unequal distribution of wealth — measured by the current state of inequality σ_t^2 and affected by the extent of redistributive activities — goes along with a lower growth rate. This is caused by the combination of imperfect capital markets, together with the concavity property of the production function. Conversely, growth could be higher in a more equal society, thereby reflecting an *opportunity–enhancing effect* (Aghion *et al.*, 1999) of redistribution. Equation (9) also illustrates that an increase in redistributive taxes results in two competing effects on growth. The first is the well–known distortionary effect on savings, which is harmful to growth. The second one is related to the opportunity–enhancing effect and also reflects the insurance effect of taxation (Domar and Musgrave, 1944; Varian, 1980). It is promoting growth, because the more equal wealth distribution from an increase in taxes ultimately results in a higher level of output, due to the concavity of the production function. Of course, we restrict parameterization of the model such that positive values of (9) are sustained in the long run and the economy evolves along a path characterized by an ongoing increase in per capita incomes.

3 The politico–economic equilibrium

By now, we have established a link between individual savings, the distribution of wealth and growth for a given tax–transfer scheme. So, the natural next step of the analysis is to discuss the interaction between these variables and the effects of redistributive politics on the economic system, if agents are allowed to vote on the degree of tax progression within a democratic process.

Empirical evidence suggests that the relatively poor engage in the democratic process less actively than the wealthier classes, see Shields and Goidel (1997, and references therein). Among others, Bénabou (1996, 2000) argues that this indicates the presence of biases in the political system which have to be taken into account in economic analysis.

Deviations from the purely democratic *one man–one vote* system can be motivated in several ways: On the one hand, one might argue that it is easier for the rich to raise funds for lobbying activities and that they face less frictions in coordinating themselves in pressure groups by building up networks. The comparably low polling of the lower income classes can then be explained with a less organized structure of interest groups, a general feeling of individual powerlessness or annoyance about political representatives, and, perhaps, with the simple explanation that individuals are more concerned with earning their living and do not actively participate in democratic processes for opportunity costs reasons. With regard to the extent of redistributive activities in the society, the underrepresentation of the poor in the political process then results in less redistribution.

On the other hand, one might also take the view that inequality aversion brings masses to raise, whenever from their point of view the perceived extent of inequality becomes too large. The economy faces the risk of socio–political instability, where the threat of being expropriated might cause the rich to support a larger amount of redistribution. Contrary, a large degree of equality, achieved by an extensive amount of redistribution and publicly provided insurance, dampens the chances of upward mobility and provides incentives to vote against redistribution.

Altogether, these arguments indicate that inequality itself might be a relevant variable in the explanation of biases in the political system, such that changes in the distribution of wealth also trigger corresponding movements in the degree of political participation. In what follows, we will assume that the position of the pivotal agent in the wealth distribution changes over time according to the extent wealth inequality evolves. In particular, it is assumed that poor people gain political influence, whenever inequality grows too large and that rich people dominate the voting process, whenever inequality is low. Redistributive cycles originate in our analysis from the endogenous determination of the wealth bias, where the distribution of political power depends on wealth inequality itself.

The underlying framework is closely related to Bénabou (1996, 2000), who also discusses consequences of deviations from the usual *one–man–one–vote* case. There the bias in the political system is taken to be time–invariant and exogenously given, an assumption which we will relax in what follows as indicated above. Moreover, our analysis differs to the respect that we assume the wealth bias in the political system to depend on the absolute level of individual wealth instead of relative wealth. Since the single agent is not concerned with his rank in the wealth distribution when voting over tax–transfer–schemes, because the absolute wealth level decides on the individually preferred degree of tax progression, it is straightforward to let the individual political weight to depend on absolute values, too.

Biased distribution of political power and inequality We start with deriving conditions on the individually preferred degree of redistribution and discuss the implications of an exogenously given wealth bias for the dynamics and long–run levels of redistribution, inequality and growth. The case of an endogenously determined wealth bias will be considered in the next section.

We assume that agents vote on the degree of tax progression in each period of time. The overlapping generations structure of the model, where parents only care about expected income instead of their offspring's expected utility, allows us to disregard strategic interactions in an intertemporal context. Otherwise, voters might have incentives to influence future political outcomes by altering the distributional dynamics via present actions.

By (1) and (6), the expected utility of agent i in period t can be determined as:

$$\mathrm{E}\left[U_t^i\right] = \overline{U}_t^i + \ln[1 - s(\tau_t)] + \gamma\beta\,\mathrm{E}[\ln k_{t+1}^i] + \gamma(1-\beta)\,\left[\mu_{t+1} + \tfrac{1}{2}\sigma_{t+1}^2\right]\ . \qquad (10)$$

Here, $\overline{U}_t^i$ collects all terms independent from τ_t, therefore being irrelevant for the subsequent analysis. By utilizing (7), (8a) and (8b), the preferred tax policy of agent i with wealth k_t^i in period t can be determined by maximizing (10) with respect to τ_t. The individually preferred degree of tax progression implicitly solves the following equation:

$$-\frac{1}{\gamma}\frac{s'(\tau_t)}{1-s(\tau_t)}+\frac{s'(\tau_t)}{s(\tau_t)}+\beta(1-\tau_t)\left[\sigma_u^2+\beta^2\sigma_t^2\right]=\beta^2[\ln k_t^i-\mu_t] \qquad (11)$$

In what follows, we consider the consequences of a distorted political system, where the bias is exogenously fixed and time invariant. Different to Bénabou (2000), we assume that the political weight ω^i of agent i depends on his absolute level of wealth.[3] If $\omega^i=(k_t^i)^\lambda$ for some $\lambda \gtrless 0$, then the pivotal voter p has (log) wealth $\ln k_t^p=\mu_t+\lambda\sigma_t^2$ (cf. Bénabou, 2000, Prop. 6, and the related proof). This means that, whenever $\lambda<0$, the pivotal agent owns less wealth than the median voter, and the political system is biased in favor of the poor. Conversely, the system displays an elitist image, if the pivotal agent is wealthier than the median, that is $\lambda>0$. For $\lambda=0$, we have the benchmark case of the standard median voter approach with $\omega^i=1$. Applying these considerations in (11), i. e. the individually preferred tax progressivity, enables us to derive the following results with respect to the political equilibrium:

Proposition 2 (Political equilibrium with exogenous wealth bias) *If the pivotal agent p has (log) wealth* $\ln k_t^p=\mu_t+\lambda\sigma_t^2$*, where* $\lambda \gtreqless 0$*, the equilibrium degree of tax progression determined in the political voting mechanism uniquely solves:*

$$\frac{1-\beta(1-\tau_t)}{(1-\tau_t)\,[1+\beta\gamma(1-\tau_t)]}-\beta(1-\tau_t)\sigma_u^2=-\beta^2\left[\lambda-\beta(1-\tau_t)\sigma_t^2\right]. \qquad (12)$$

The equilibrium tax rate τ_t *depends on the current level of inequality* σ_t^2 *and the wealth bias* λ *according to the function* $\tau_t\equiv T(\sigma_t^2,\lambda)$*, displaying the following properties:*

(i) The equilibrium rate $T(\sigma_t^2,\lambda)$ *is strictly decreasing in* λ*, i. e.* $T_\lambda(\sigma_t^2,\lambda)<0$.

(ii) The effect of inequality on the degree of tax progression is ambiguous, i. e. $T_{\sigma^2}(\sigma_t^2,\lambda)\gtreqless 0$*, where the sign of the derivative depends on the degree of distortion in the political system. In the special case of an unbiased voting mechanism (i. e.* $\lambda=0$*), we have* $T_{\sigma^2}(\sigma_t^2,0)>0$.

Proof. See Appendix. □

Thus, the interaction between redistribution and inequality depends on the extent of the wealth bias in the political system. Whenever the bias λ is

[3] This assumption is not essential for the cyclical dynamics of the model, which would also appear, if we assumed the political weight to depend on relative wealth.

not too large — in particular in the unbiased median voter case of $\lambda = 0$ — the volume of redistribution increases with inequality.[4]

If, contrary, the bias in favor of the rich is sufficiently large (only for $\lambda \gg 0$), more inequality can lead to less redistribution. This means that for a considerably uneven distribution of political power, we have a pivotal voter who is rich enough to prefer less redistribution as inequality rises.

With $\tau_t = T(\sigma_t^2, \lambda)$ as the outcome of the political process, the dynamics of wealth inequality as derived in (8b) are now described by the following nonlinear difference equation:

$$\sigma_{t+1}^2 = \sigma_z^2 + [1 - T(\sigma_t^2, \lambda)] (\beta^2 \sigma_t^2 + \sigma_u^2) . \qquad (13)$$

Regarding issues of existence and uniqueness of stationary solutions of (13) we find that these crucially depend on the size of λ. We can state the following:

Proposition 3 (Existence and uniqueness of stationary solutions)

(i) *For* $\sigma_u^2 \leq \beta^2 \sigma_z^2$, *there exists a unique stationary solution* $\sigma_*^2(\lambda)$ *to* (13) *iff* $\lambda < 1$. *This solution implies* $\partial \sigma_*^2 / \partial \lambda > 0$ *with* $\lim_{\lambda \to 1} \sigma_*^2(\lambda) = \infty$.

(ii) *For* $\sigma_u^2 > \beta^2 \sigma_z^2$, *there exists a unique stationary solution* $\sigma_*^2(\lambda)$ *to* (13) *for all* $\lambda \leq 1$. *There exists an upper bound* $\overline{\lambda} > 1$, *such that there are two stationary solutions to* (13) *for* $\lambda \in (1, \overline{\lambda})$. *With respect to the multiple solutions, one is characterized by more redistribution and less inequality than the other. No stationary solution* $\sigma_*^2(\lambda)$ *exists for* $\lambda > \overline{\lambda}$.

Proof. See Appendix. □

Figure 1 shows, how the stationary level of inequality varies with the bias of the political system λ in the two cases described in Proposition 3. The relation between the uninsurable and the insurable risk is crucial for the emergence of multiple equilibria. The higher the uninsurable risk, the more likely multiple equilibria occur. A value of $\lambda = 1$ has the easy interpretation of reflecting the *one–dollar–one–vote* case.

Regarding the dynamic properties of the model for a given level of the wealth bias λ, we can state the following:

[4]The median voter outcome is well–known from the literature. Meltzer and Richard (1981) were the first to argue within a general equilibrium context that, if the underlying distribution is right–skewed (e. g. here the lognormal distribution), the median voter always prefers a positive amount of redistribution. Redistributive activities increase the poorer the median is compared to the mean.

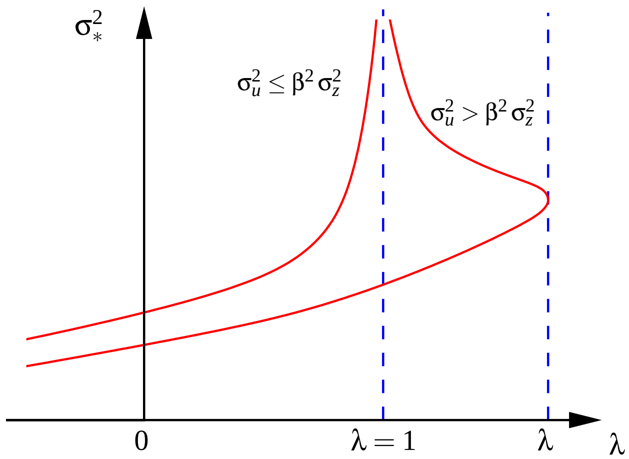


Figure 1: *Stationary levels of inequality*

Proposition 4 (Stability of stationary solutions) *For* $\sigma_u^2 \leq \beta^2 \sigma_z^2$*, the unique stationary solution to* (13) *is always stable. Whenever* $\sigma_u^2 > \beta^2 \sigma_z^2$*, the stationary solution to* (13) *associated with lower inequality is stable, whereas the solution associated with higher inequality is unstable.*

Proof. See Appendix. □

That the dynamical system might display multiple equilibria in case of an exogenously fixed political bias is a well–known feature from the analysis of biased political systems (cf. Bénabou, 2000), but of minor importance for our argument, since we are primarily interested in solutions which are consistent with convergence to a stable equilibrium. For this reason, we exclude the unstable equilibrium associated with a higher level of inequality and a lower degree of tax progression from our discussion.

Endogenous cycles In order to formalize the idea of cyclical behavior of the economic system, we now assume that the political bias λ_t depends on the degree of inequality, measured by the variance of wealth. The underlying law of motion is given by $\lambda_{t+1} = H(\sigma_t^2)$. By this we posit that the extent to which the future political system deviates from the 'ideal' of the median voter equilibrium is determined by the current level of wealth inequality. With respect to the function $H(\sigma^2)$, we furthermore assume that the value of H declines with a growing variance of wealth, $H'(\sigma^2) < 0$, thus capturing the idea that an increase in inequality shifts the political power towards the

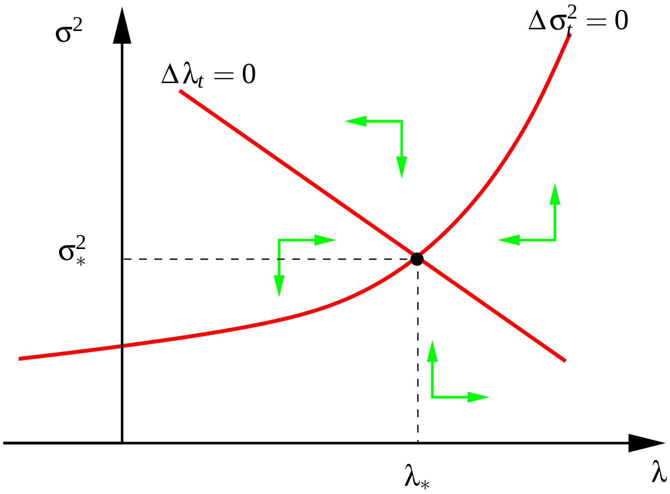


Figure 2: *Phase diagram in the presence of an endogenous political bias*

poor.[5] The modifications now lead us to a two–dimensional system, jointly describing the evolution of wealth inequality σ_t^2 and the political bias λ_t:

$$\sigma_{t+1}^2 = \sigma_z^2 + \left[1 - T(\sigma_t^2, \lambda_t)\right]^2 \left(\sigma_u^2 + \beta^2 \sigma_t^2\right) , \tag{14a}$$

$$\lambda_{t+1} = H(\sigma_t^2) . \tag{14b}$$

Figure 2 shows the associated phase diagram of the dynamical system. It combines the information on the dynamics of inequality available from Proposition 4 with the properties of $H(\sigma_t^2)$ as stated above. For illustrative purposes, we established a simple linear relationship between the wealth bias and inequality.[6]

Let us now analyze the dynamics of the model in the neighborhood of the stationary point (σ_*^2, λ_*). Recall from our previous discussion of an exogenously fixed political bias that, as we now extend the analysis to an endogenously determined λ, the argument is restricted only to those fixed points which were shown to be stable for λ being held constant. Let the parameter $h = H'(\sigma_*^2) < 0$ denote the partial derivative of the political bias with respect to inequality, evaluated at the stationary state. The dynamics of the system (14a) and (14b) are then characterized as follows:

Proposition 5 (Hopf–bifurcation) *Let $H'(\sigma_*^2) = h$, with h being a parameter. Then there exists a value $h^b < 0$, such that*

---

[5] Accordingly, $H'(\sigma^2) = 0$ reflects the above discussed case of an exogenous wealth bias.

[6] A linear functional form of $H(\sigma^2)$ is sufficient for our argument, since the local dynamics (i. e. the linearized system) are relevant in order to show the emergence of cycles.

(i) σ^2_*, λ_* *is a stable stationary solution of the system* (14a) *and* (14b) *for all* $h > h^b$*,*

(ii) the dynamical system undergoes a Hopf–bifurcation at the bifurcation value h^b*.*

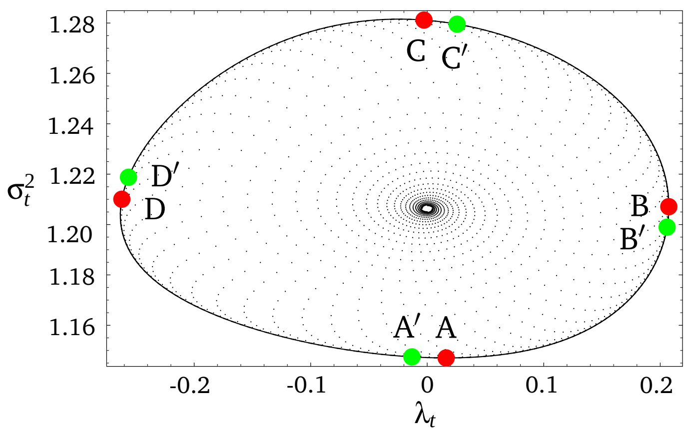
Proof. See Appendix. □

Proposition 5 establishes the existence of cycles in our model. Since the stability properties of the stationary point (σ^2_*, λ_*) depend on the value of h, this represents a bifurcation parameter. As the phase diagram of Figure 2 already indicates, there exists a stable closed curve in the neighborhood of the (now) unstable stationary point (σ^2_*, λ_*) for values $h < h^b$. This closed curve then represents an endogenous cycle of political participation, redistribution and growth. The numerical simulations provided in Figure 3 below demonstrate that the Hopf–bifurcation is supercritical.

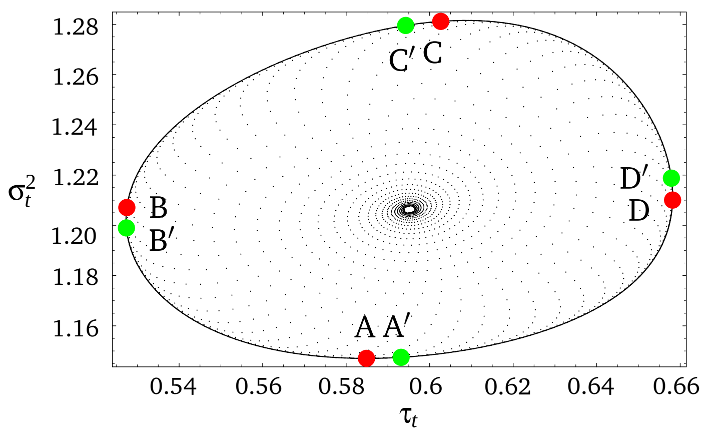
In these simulations we postulate a simple linear relationship for the dynamics of political participation, $\lambda_{t+1} = h\,(\sigma^2_t - \sigma^2_*)$, such that $\lambda_* = 0$, and set the parameters according to $\beta = 0.98$, $\gamma = 5$, $A = 1.5527$, $\sigma^2_u = 0.1$ and $\sigma^2_z = 1$. This implies a value of $\sigma^2_* = 1.20629$ for long–run wealth inequality and an associated elasticity of post–tax investment of $\tau_* = 0.595135$ in the median voter equilibrium ($\lambda_* = 0$), together with an empirically plausible mean income growth rate of $g_{y_*} = 0.02$ and an equilibrium value for the Gini coefficient of around 0.56. The dynamical system now undergoes the Hopf–bifurcation for a value of $h = h^b = -3.47103$. The simulation results presented in Figure 3 are plotted for an arbitrarily chosen slope of $h = -3.5$, satisfying the condition $h < h^b$.

Figure 3(a) plots the cyclical interaction between wealth inequality and political participation in the σ^2_t/λ_t–plane, while Figure 3(b) displays the corresponding link between inequality and redistributive politics in the σ^2_t/τ_t–plane, and, finally, Figure 3(c) showing the tradeoff relationship between growth and inequality in the $\sigma^2_t/g_{y_{t+1}}$–plane. As becomes obvious, periods of low growth due to a large amount of redistribution go along with a political bias favoring the poor. In the course of decreasing wealth inequality, the political power shifts towards the rich, who enforce tax–transfer–schemes entailing a low degree of redistribution and larger growth rates. This causes inequality to rise, thereby initiating a backward shift of power to the poor.

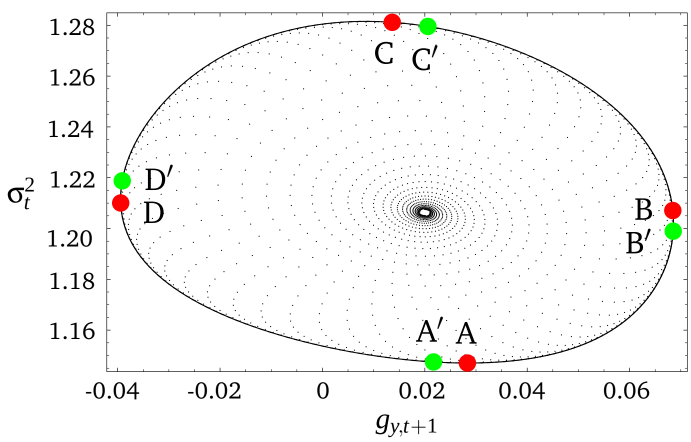
For an easier understanding, we have tagged two successive cycles on the closed curve, the first represented by the four consecutive points A, B, C, the second given by A', B', C', and D'. Consider first the cyclical dynamics



(a) Inequality and political participation



(b) Inequality and tax progression



(c) Inequality and growth

Figure 3: *Cyclical dynamics of income inequality, redistribution and growth*

of wealth inequality and the bias in political participation. As indicated by the dynamic system (14a) and (14b), the political bias will shift towards the rich, whenever inequality is below its stationary value and *vice versa*, whereas the opposite holds for changes in inequality, which rises, if the political bias is above its stationary level of $\lambda_* = 0$ and declines otherwise (see also Figure 2).

Point *A* in Figure 3(a) reflects a situation, where inequality is at its possibly lowest level, while the bias approximately attains its stationary level. Consequently, wealth inequality as well as the political power of the rich increase throughout the transition from *A* to *B*. Although inequality almost reaches its stationary value in *B*, the political bias now exceeds λ_*. In the next instance, while wealth inequality rises even further, the bias already declines. This indicates the transition from *B* to *C*. Point *C* forms the antipode to *A*, again reflecting a political bias being close to its stationary value, whereas inequality now exceeds the level of σ^2_*. This causes participation dynamics to shift the political power towards the poor, leading to a decline in inequality in point *D*. Now, the bias in political participation is below its stationary value, indicating further decreases in inequality, which causes the political bias to rise again. The economy then moves to A', the starting point of the subsequent cycle which passes through B', C' and D'.

The cyclical behavior of redistributive politics and mean income growth follows naturally. If we look at the transition from *A* to *B* — the first characterized by low inequality and a (almost) median voter outcome, the second by larger inequality and a strong political bias towards the rich — we observe that redistributive activities decline. The extent of redistribution is determined by two counter–acting effects. On the one hand, the increase in inequality supports higher taxation. On the other hand, the shift in the political bias lets the rich dominate the voting outcome, which works against redistribution. The latter effect outweighs the first, implying a lower degree of tax progression which is accompanied by an increase in the growth rate of mean income. Here, the positive incentive effect on the savings rate due to less tax progressivity dominates the negative growth effect from an increase in inequality due to the presence of imperfect capital markets, recall eq. (9). Throughout the cycle the growth rate varies between −4% and 7% and taxation is progressive over the entire cycle.[7]

[7]The lower bound of a feasible degree of tax progression for the given parameterization of the model can be determined as $\underline{\tau} = -0.02$. This does not exclude the possibility that, for alternative specifications of the model primitives, the system might also undergo phases of regressive taxation.

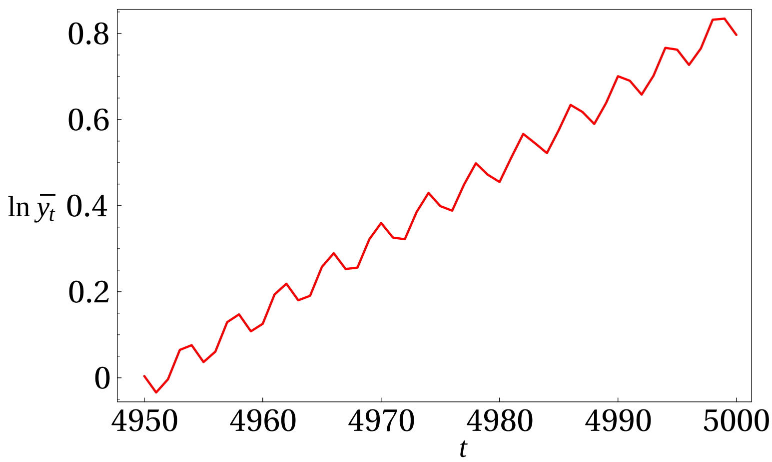


Figure 4: *Time path of mean income $\bar{y}_t$*

Figure 4 finally shows the time path of mean income over an arbitrarily chosen time span of 50 generations. The growth process is cyclical with a frequency of four generations and possesses a non–regular amplitude. For instance, cycles can display one–period recessions as well as recessions lasting for two periods. The peaks reflect a comparably 'elitist' society, whereas the troughs correspond to a more 'populist' system.

Having so far established the possible emergence of redistributive cycles, a natural next step is to investigate how sensitive the model dynamics are to changes in the bifurcation parameter h, measuring the magnitude to which the bias in political participation responds to changes in wealth inequality.

Figure 5 shows a bifurcation diagram of the model with respect to wealth inequality. The numerical specification of the model is the same as used in the above described simulations. The system (14a) and (14b) was simulated over 7500 periods for different values of h, starting with values σ_0^2 and λ_0 close to the stationary point σ_*^2, λ_* and taking a step size for h of $\Delta h = 0.025$. The first 5000 values for σ_t^2 are dropped in order to exclude effects from transitory dynamics.

The Figure shows the values of inequality which are visited during the last 2500 periods. As can be seen, a stable stationary solution exists for small values $h > h^b$. The supercritical Hopf–bifurcation generates a stable closed curve at the value $h^b \approx -3.47103$. The bifurcation diagram also reveals the non–regular nature of the cycles in this area of numerical values for h. Each point on the closed curve is visited as time moves on.[8] If we reduce h further

[8]See the discussion of the two consecutive cycles A to D and A' to D' related to Figure 3.

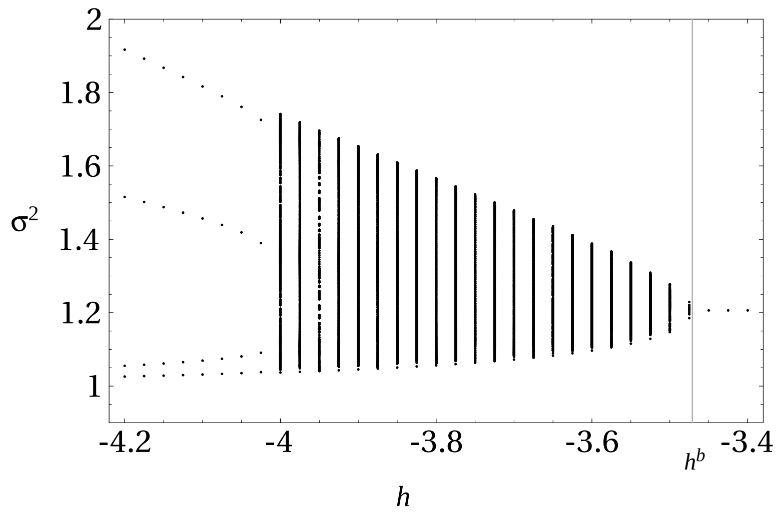


Figure 5: *Bifurcation diagram for inequality* σ^2

below the bifurcation value h^b, the dynamic behavior of the model changes again, giving rise to regular cycles with a frequency of four periods. The response of the political bias to changes in the extent of inequality is comparably strong, such that the economy is trapped in four states which are repeatedly taking turns. The twofold switch in the dynamic behavior of the system — first from a stable stationary state to cyclical movements and second from non–regular to regular cycles — is robust to changes in the model primitives.

4 Concluding Remarks

In this paper, we investigated a model of stochastic growth with incomplete asset markets and heterogeneous agents, where redistributive cycles and cyclical growth emerge as outcomes of voting processes over public tax–transfer schemes. Heterogeneity among agents stems from idiosyncratic risks. The members of the society decide *ex ante* on the implementation of a redistributive scheme. This, consequently, serves the simple purpose of providing an insurance against unfavorable outcomes of current individual income shocks, whereas we assumed future risk to be non–diversifiable. Redistribution is not costlessly available. Costs accrue endogenously from the redistributive process in terms of disincentives to save and subsequently forgone growth.

Redistributive cycles emerge for the case of an uneven distribution of political power, in particular, if we tie the degree of political participation to inequality itself. Supported by empirical evidence, it is sufficient to postulate a negative relationship between the bias in the political system and wealth inequality in order to establish cyclical behavior of redistributive activities and growth.

From the economic point of view, the agents face two counter–acting forces. On the one hand, whenever inequality grows too large, inequality (and risk) aversion leads to a larger extent of redistribution. On the other hand, the equalizing effects stemming from a comparably high tax progression dampen individual prospects of upward mobility. This induces a shift in the bias of political power towards the relatively rich, thereby causing more and more agents to vote for a lower degree of progressivity in taxation. This is accompanied by less redistribution, and lasts, until inequality again has grown to an extent, where inequality aversion dominates the voting equilibrium and a more progressive tax–transfer–scheme is reestablished. Since redistribution provides negative incentives for individual saving, we also observe a non–linear pattern of saving and growth rates over the political cycle.

From a technical point of view, the dynamic system undergoes a supercritical Hopf–bifurcation, thereby allowing for the emergence of cyclical behavior for an appropriate value of the bifurcation parameter, which here measures the response of the bias in political power to changes in wealth inequality. Up to now, this response is exogenous to the model. So it might be a worthwhile extension of the preceding framework to endogenize the established link between political participation and inequality by making it subject to individual optimization. However, this issue is beyond the scope of the present paper.

Simulations of the numerically specified model show that our approach is capable of replicating growth rates in an empirically plausible range. The economy passes through a cycle in four generations in our simulations, which is not too overwhelming, if one takes 'generations' literally, but becomes more convincing, if one is willing to focus on the average duration of electoral cycles in modern democracies. Nevertheless, the model has to pass the empirical test, which also is left for future work.

References

Acemoglu, Daron and Robinson, James (2000), Why Did the West Extend the Franchise? Democracy, Inequality and Growth in a Historical Perspective, *Quarterly Journal of Economics*, **115**, 1167–1199.

Ades, A. and Verdier, Thierry (1996), The Rise and the Fall of Elites: Economic Development and Social Polarization in Rent–Seeking Economies, Discussion Paper 1495, Centre for Economic Policy Research (CEPR), London.

Aghion, Philippe and Bolton, Patrick (1992), Distribution and Growth in Models of Imperfect Capital Markets, *European Economic Review*, **36**, 603–611.

Aghion, Philippe, Caroli, Eve, and García-Peñalosa, Cecilia (1999), Inequality and Growth: The Perspective of the New Growth Theories, *Journal of Economic Literature*, **37**, 1615–1660.

Alesina, A. and Rodrik, Dani (1994), Distributive Politics and Economic Growth, *Quarterly Journal of Economics*, **109**, 465–490.

Azariadis, Costas (1993), *Intertemporal Macroeconomics*, Blackwell Publishers, Oxford.

Banerjee, Abhijit V. and Newman, Andrew F. (1991), Risk–bearing and the Theory of Income Distribution, *Review of Economic Studies*, **58**, 211–235.

Becker, Gary S. and Tomes, Nigel (1979), An Equilibrium Theory of the Distribution of Income and Intergenerational Mobility, *Journal of Political Economy*, **87**, 1153–1189.

Bénabou, Roland (1996), Inequality and Growth, in: Ben S. Bernanke and Julio J. Rotemberg (eds.), *NBER Macroeconomics Annual*, MIT Press, Cambridge, MA, vol. 11, pp. 11–74.

— (2000), Unequal Societies: Income Distribution and the Social Contract, *American Economic Review*, **90** (1), 96–129.

Bénabou, Roland and Ok, Efe A. (2001), Social Mobility and the Demand for Redistribution: The POUM Hypothesis, *Quarterly Journal of Economics*, **116**, 447–487.

Benhabib, Jess and Rustichini, Aldo (1996), Social Conflict and Growth, *Journal of Economic Growth*, **1** (1), 129–146.

Bertola, Giuseppe (1993), Factor Shares and Savings in Endogenous Growth, *American Economic Review*, **83**, 1185–1198.

Bourguignon, François and Verdier, Thierry (2000), Oligarchy, Democracy, Inequality and Growth, *Journal of Development Economics*, **62**, 285–313.

Corneo, Giacomo and Grüner, Hans Peter (2000), Social Limits to Redistribution, *American Economic Review*, **90** (5), 1491–1507.

Domar, Evsey D. and Musgrave, Richard A. (1944), Proportional Income Taxation and Risk–Taking, *Quarterly Journal of Economics*, **59**, 388–422.

Drazen, Allan (2000), *Political Economy in Macroeconomics*, Princeton University Press, Princeton.

Feldstein, Martin S. (1969), The Effects of Taxation on Risk Taking, *Journal of Political Economy*, **77**, 755–764.

Galor, Oded and Zeira, Joseph (1993), Income Distribution and Macroeconomics, *Review of Economic Studies*, **60**, 35–52.

Goodwin, R. M. (1951), The Nonlinear Accelerator and the Persistence of Business Cycles, *Econometrica*, **19**, 1–17.

Gradstein, Mark and Justman, Moshe (1995), A Political Interpretation of the Kuznets Curve, Working paper, Ben–Gurion University, Tel Aviv.

Kaldor, Nicholas (1940), A Model of Trade Cycle, *Economic Journal*, **50**, 78–92.

Kanbur, S. M. (1979), Of Risk Taking and the Personal Distribution of Income, *Journal of Political Economy*, **87**, 769–797.

Krusell, Per and Smith, Anthony A. (1998), Income and Wealth Heterogeneity in the Macroeconomy, *Journal of Political Economy*, **106** (5), 867–896.

Krusell, Per, Quadrini, Vincenzo, and Ríos-Rull, José-Víctor (1997), Politico–economic Equilibrium and Economic Growth, *Journal of Economic Dynamics and Control*, **21**, 243–272.

Lindert, Peter H. (1996), What Limits Social Spending?, *Explorations in Economic History*, **33**, 1–34.

Loury, Glenn C. (1981), Intergenerational Transfers and the Distribution of Earnings, *Econometrica*, **49**, 843–867.

Matsuyama, Kiminori (2000), Endogenous Inequality, *Review of Economic Studies*, **67**, 743–759.

Meltzer, Allan H. and Richard, Scott F. (1981), A Rational Theory of the Size of Government, *Journal of Political Economy*, **89**, 914–927.

Mirrlees, James A. (1971), An Exploration in the Theory of Optimal Income Taxation, *Review of Economic Studies*, **38**, 175–208.

Perotti, Roberto (1993), Political Equilibrium, Income Distribution, and Growth, *Review of Economic Studies*, **60**, 755–776.

Persson, Torsten and Tabellini, Guido (1994), Is Inequality Harmful for Growth?, *American Economic Review*, **84** (3), 600–621.

Piketty, Thomas (1995), Social Mobility and Redistributive Politics, *Quarterly Journal of Economics*, **110**, 551–584.

— (1997), The Dynamics of Wealth Distribution and the Interest Rate with Credit Rationing, *Review of Economic Studies*, **64**, 173–189.

Plümper, Thomas and Martin, Christian W. (2003), Democracy, Government Spending, and Economic Growth: A Political–economic Explanation of the Barro–effect, *Public Choice*, **117**, 27–50.

Romer, Paul M. (1986), Increasing Returns and Long–Run Growth, *Journal of Political Economy*, **94**, 1002–1037.

Saint-Paul, Gilles and Verdier, Thierry (1993), Education, Democracy and Growth, *Journal of Development Economics*, **42** (2), 399–407.

— (1997), Power, Distributive Conflicts, and Multiple Growth Paths, *Journal of Economic Growth*, **2**, 155–168.

Shields, Todd G. and Goidel, Robert K. (1997), Participation Rates, Socioeconomic Class Biases and Congressional Elections: A Crossvalidation, *American Journal of Political Science*, **41** (2), 683–691.

Varian, Hal R. (1980), Redistributive Taxation as Social Insurance, *Journal of Public Economics*, **14**, 49–68.

Appendix

Derivation of the difference equation for (log) wealth By eqs. (3) and (5), individual (log) wealth evolves according to the following difference equation:

$$\ln k^i_{t+1} = \ln z^i_t + \ln \tilde{x}^i_t = \ln z^i_t + (1-\tau_t)\ln x^i_t + \tau_t \ln \tilde{x}_t \tag{A.1}$$

In order to derive the break–even level of (log) investment $\ln \tilde{x}_t$, let $\mu_{y,t}$ and $\sigma^2_{y,t}$ denote the first and second moments of the distribution of (log) income. By (4) and (6), we get

$$\ln \int_0^1 x^i_t \, \mathrm{d}i = \ln s(\tau) + \mu_{y,t} + \sigma^2_{y,t}/2$$

as well as

$$\ln \int_0^1 \tilde{x}^i_t \, \mathrm{d}i = \ln \left(\int_0^1 {x^i_t}^{1-\tau} \tilde{x}^\tau_t \, \mathrm{d}i \right) = (1-\tau)[\ln s(\tau) + \mu_{y,t}] + (1-\tau)^2 \sigma^2_{y,t}/2 + \tau \ln \tilde{x}_t \, .$$

Equating the RHS of both equations, yields the following expression for the break–even level of investment $\ln \tilde{x}_t$:

$$\ln \tilde{x}_t = \ln s(\tau) + \mu_{y,t} + (2-\tau)\frac{\sigma^2_{y,t}}{2} \, .$$

From $\ln y^i_t = \ln A + \ln u^i_t + \beta \ln k^i_t + (1-\beta)\ln \kappa_t$ follows with (log) average wealth given by $\ln \kappa_t = \mu_t + \sigma^2_t/2$:

$$\mu_{y,t} = \ln A - \frac{\sigma^2_u}{2} + \mu_t + (1-\beta)\frac{\sigma^2_t}{2} \, ,$$
$$\sigma^2_{y,t} = \sigma^2_u + \beta^2 \sigma^2_t \, .$$

Substituting these expressions into the definition of $\ln \tilde{x}_t$ leads to:

$$\ln \tilde{x}_t = \ln s(\tau_t) + \ln A + \mu_t + (1-\tau_t)\frac{\sigma^2_u}{2} + \left[1 - \beta + \beta^2(2-\tau_t)\right]\frac{\sigma^2_t}{2} \, .$$

Inserting this expression into (A.1) finally implies equation (7) of the text. □

Proof of Proposition 2: After substitution of the savings rate $s(\tau_t)$ into the first order condition (11), we define the function

$$G(\tau_t) \equiv \frac{1-\beta\,(1-\tau_t)}{(1-\tau_t)\,[1+\beta\gamma(1-\tau_t)]} - \beta\,(1-\tau_t)\,\sigma_u^2\,.$$

The necessary condition for the individually optimal rate τ_t can now be rewritten as

$$G(\tau_t) = \beta^3\,(1-\tau_t)\,\sigma_t^2 + \beta^2\left[\mu_t - \ln k_t^i\right]\,. \tag{A.2}$$

$G(\tau)$ is strictly increasing and convex in τ_t with $G(\tau)\to\infty$ for $\tau\to 1$ and $G(\tau)\to-\infty$ for $\tau\to-\infty$. In what follows, $\underline{\tau} = 1-1/\beta$ denotes the smallest long–run tax rate to be consistent with a stationary level of inequality (see footnote 2 on page 6). This implies $G(\underline{\tau}) = -\sigma_u^2$ and $G'(\underline{\tau}) = \frac{\beta^2}{1+\gamma} + \beta\sigma_u^2$. Let now $\underline{\tau} < \tilde{\tau} < 1$ denote the unique root of $G(\tau)$.

The properties of $G(\tau)$ imply that preferences of agents are single peaked over τ_t such that there exists a unique solution to equation (A.2) for each level of individual wealth k_t^i. If we now assume that the pivotal individual has (log) wealth $\ln k_t^p = \mu_t + \lambda\sigma_t^2$, the tax rate of the political equilibrium is the solution to

$$G(\tau_t) = -\beta^2\,[\lambda - \beta(1-\tau_t)]\,\sigma_t^2\,. \tag{A.3}$$

For λ given, the RHS of (A.3) is a linear function in τ_t, crossing the horizontal axis at $\tau_t = 1-\lambda/\beta$, where increasing values of σ_t^2 imply a clockwise rotation of this function in the intersection point. (A.3) is solved at the intersection of the nonlinear function $G(\tau_t)$ with the linear RHS of (A.3). Given the properties of $G(\tau_t)$, this finally implies that the equilibrium tax rate in each period depends on inequality σ_t^2 and the wealth bias λ according to a function $\tau_t = T(\sigma_t^2,\lambda)$, the latter characterized by the following properties (see also Figure 6 for an illustration based on two arbitrarily chosen levels λ_0 and λ_1):

(T.1) If $\lambda \le 0$, we have $T_{\sigma^2}(\sigma_t^2,\lambda) > 0$ with $T(\sigma_t^2,\lambda)\to\tilde{\tau}$ for $\sigma_t^2\to 0$ and $T(\sigma_t^2,\lambda)\to 1$ for $\sigma_t^2\to\infty$.

(T.2) If $0<\lambda<\beta\,(1-\tilde{\tau})$, we have $T_{\sigma^2}(\sigma_t^2,\lambda) > 0$ with $T(\sigma_t^2,\lambda)\to\tilde{\tau}$ for $\sigma_t^2\to 0$ and $T(\sigma_t^2,\lambda)\to 1-\lambda/\beta$ for $\sigma_t^2\to\infty$.

(T.3) If $\lambda = \beta\,(1-\tilde{\tau}) > 0$, we have $T(\sigma_t^2,\lambda) = \tilde{\tau}$ for all $\sigma_t^2 \ge 0$.

(T.4) If $\beta\,(1-\tilde{\tau}) < \lambda < 1$, we have $T_{\sigma_t^2}(\sigma^2,\lambda) < 0$ with $T(\sigma_t^2,\lambda)\to\tilde{\tau}$ for $\sigma_t^2\to 0$ and $T(\sigma_t^2,\lambda)\to 1-\lambda/\beta < \underline{\tau}$ for $\sigma_t^2\to\infty$.

(T.5) If $\lambda = 1$, we have $T_{\sigma_t^2}(\sigma^2,\lambda) < 0$ with $T(\sigma_t^2,\lambda)\to\tilde{\tau}$ for $\sigma_t^2\to 0$ and $T(\sigma_t^2,\lambda)\to\underline{\tau}$ for $\sigma_t^2\to\infty$.

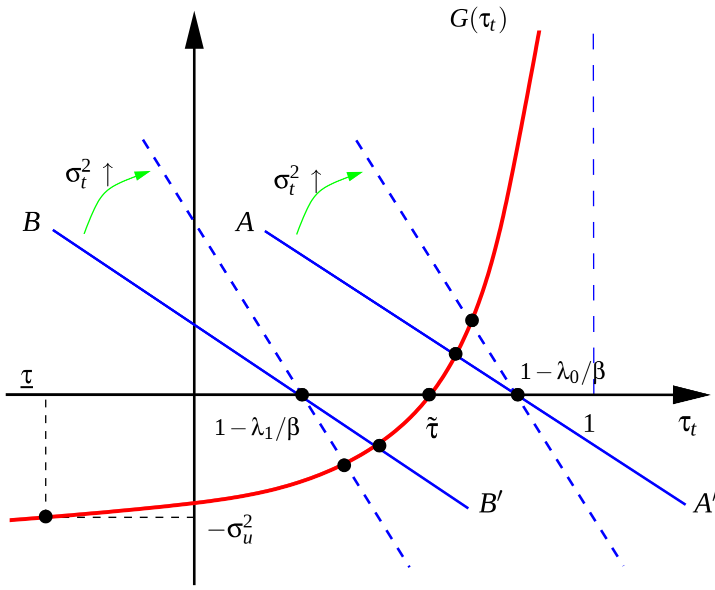
(T.6) If $\lambda > 1$, we have $T_{\sigma^2}(\sigma_t^2,\lambda) < 0$ with $T(\sigma_t^2,\lambda)\to\tilde{\tau}$ for $\sigma_t^2\to 0$ and $T(\sigma_t^2,\lambda)\to 1-\lambda/\beta < \underline{\tau}$ for $\sigma_t^2\to\infty$.

(T.7) $T(\sigma^2,\lambda)$ is strictly decreasing in λ, i. e. for a given level of inequality σ^2, we have $T_\lambda(\sigma^2,\lambda) < 0$.

□

Proof of Proposition 3: Let τ_* denote the stationary tax rate. According to (8b), the stationary level of inequality $\sigma_*^2 \equiv S(\tau_*)$ is a function in τ_* and can be determined as

$$S(\tau_*) = \frac{\sigma_z^2 + (1-\tau_*)\,\sigma_u^2}{1-\beta^2\,(1-\tau_*)^2} \tag{A.4}$$



With $\beta(1-\tilde{\tau}) > \lambda_0 > 0$ (i. e. case (T.2)), the RHS of eq. (A.2) is given by the line AA' with slope $-\beta^3\sigma_t^2$ crossing the abscissa at $1-\lambda_0/\beta$. An increase in σ_t^2 leads to a clockwise rotation of this line. Since $1-\lambda_0/\beta > \tilde{\tau}$, the equilibrium tax rate increases, too. For $\sigma_t^2 \to \infty$ we have $\tau \to 1-\lambda_0/\beta$, while $\tau \to \tilde{\tau}$ for $\sigma_t^2 \to 0$. The value λ_1 (i. e. case (T.4)) implies that $1-\lambda_1/\beta > \tilde{\tau}$. Now, a clockwise rotation of the line BB' goes along with a decrease of the equilibrium tax rate.

Figure 6: *Properties of* $T(\sigma_t^2, \lambda)$

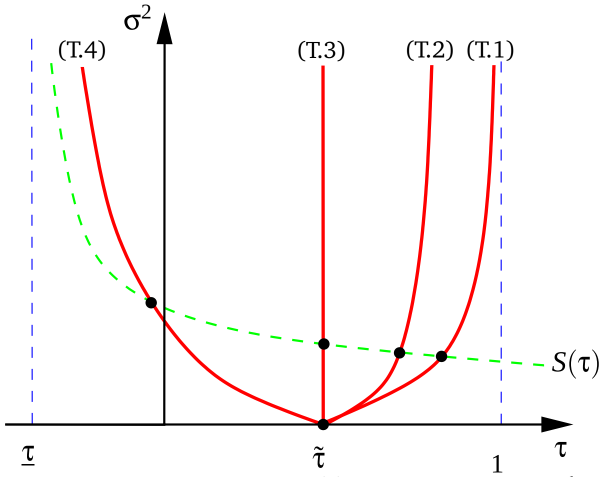
which requires $\underline{\tau} < \tau_* < 1$. We also know from Proposition 2 that the stationary rate τ_* depends on inequality via the function $\tau_* = T(\sigma_*^2, \lambda)$. This implies that a stationary solution for inequality is characterized by the intersection of the function $S(\tau_*)$ with the function $T(\sigma_*^2, \lambda)$. Together with the above discussed properties (T.1) – (T.4) of $T(\sigma^2, \lambda)$, this immediately proves the existence of a unique stationary solution, for $\lambda < 1$ (see also Figure 7(a) for a graphical illustration). $S(\tau_*)$ increases with λ, which follows from the observation that, by (A.4), the stationary level of inequality $S(\tau_*)$ rises as τ_* decreases. Additionally, according to Proposition 2, τ_* decreases for an increase in λ.

An open question is, whether or not there exists an interior solution for $\lambda = 1$. In order to analyze this case, we insert the stationary value of inequality $S(\tau)$ into the right hand side of equation (A.3) and define the resulting expression with $M(\tau) \equiv -\beta^2\,[1-\beta\,(1-\tau)]\,\frac{\sigma_z^2+(1-\tau)\,\sigma_u^2}{1-\beta^2\,(1-\tau)^2}$. The existence of a stationary solution for $\lambda = 1$ now demands a feasible tax rate to solve $G(\tau) = M(\tau)$ with $\underline{\tau} < \tau < 1$. With respect to $M(\tau)$ we obtain:

$$M(\underline{\tau}) = -\frac{\beta^2\,\sigma_z^2 + \sigma_u^2}{2} \tag{A.5a}$$

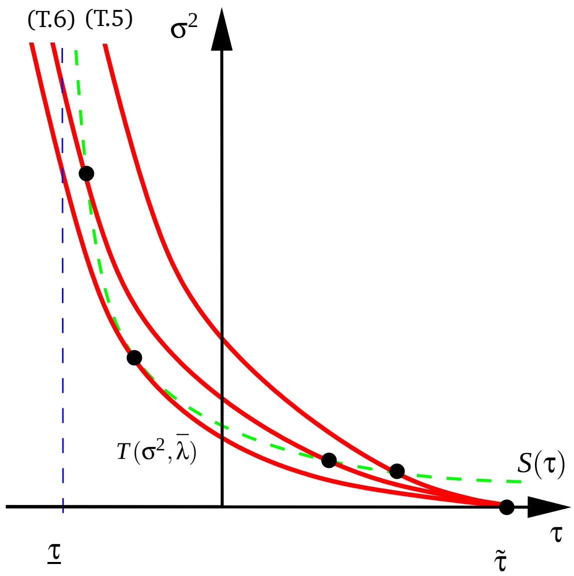
$$M'(\underline{\tau}) = \frac{3\,\beta\sigma_u^2 - \beta^3\,\sigma_z^2}{4}, \qquad M''(\tau) = -\frac{2\,\beta^2\,(\sigma_u^2 + \beta^2\,\sigma_z^2)}{[1+\beta\,(1-\tau)]^3} < 0 \tag{A.5b}$$

Notice, that (A.5b) implies that $M(\tau)$ is a concave function, with $M'(\underline{\tau})$ being strictly smaller than $G'(\underline{\tau})$. Hence, no interior solution of $G(\tau) = M(\tau)$ exists, whenever $G(\underline{\tau}) \geq M(\underline{\tau})$, where this last inequality is equivalent to $\sigma_u^2 \leq \beta^2\sigma_z^2$.



(a) Unique stationary solution

The Figure shows the function $S(\tau)$ together with functions $T(\sigma^2,\lambda)$ for each of the differentiated cases (T.1)–(T.4) described in the proof of Proposition 2. The existence of a unique stationary solution for $\lambda < 1$ follows accordingly.



(b) Multiple stationary solutions in case $\sigma_u^2 > \beta^2\sigma_z^2$

If $\sigma_u^2 > \beta^2\sigma_z^2$, there exists a unique intersection of $T(\sigma^2,1)$ with $S(\tau)$. Therefore, multiple solutions must exist for $\lambda > 1$. For $\lambda = \bar{\lambda}$, the function $T(\sigma^2,\lambda)$ is tangent to $S(\tau)$.

Figure 7: *Stationary solutions*

If $\sigma_u^2 > \beta^2\sigma_z^2$, we have $G(\underline{\tau}) < M(\underline{\tau})$, such that there exists an interior solution (with $\tau > \underline{\tau}$) for $G(\tau) = M(\tau)$. This implies a unique intersection of the function $S(\tau_*)$ with the function $T(\sigma_*^2, 1)$, yielding a solution $\underline{\tau} < \tau_* < 1$.

Moreover, from $T(\sigma^2,\lambda)$ being strictly decreasing in λ follows that there exists an open set $(1,\bar{\lambda})$, with $\bar{\lambda} > 1$, such that we also obtain feasible solutions to the two equations $S(\tau_*)$ as well as $T(\sigma_*^2,\lambda)$ in the interval $\lambda \in (1,\bar{\lambda})$. Since $S(\tau) \to \infty$ as $\tau \to \underline{\tau}$, while $T(\sigma^2,\lambda) \to 1-\lambda/\beta < \underline{\tau}$ as $\sigma^2 \to \infty$ we find exactly two such solutions for $\lambda \in (1,\bar{\lambda})$ (see also 7(b) for a graphical illustration). □

Proof of Proposition 4: The proof follows the proof of Theorem 1 in Bénabou (2000). As is proven there, a stationary solution (σ_*^2, τ_*) is stable, whenever the curve $\sigma^2 = S(\tau)$ cuts the curve $\tau = T(\sigma^2, \lambda)$ from above at (σ_*^2, τ_*).

From Proposition 2 then follows that in case of $\sigma_u^2 \leq \beta^2 \sigma_z^2$ this condition is met and the unique stationary solution is always stable. In case of $\sigma_u^2 > \beta^2 \sigma_z^2$, a unique and stable stationary solution exists for identical reasons for all $\lambda \leq 1$. In case of multiple stationary solutions, i. e. $\lambda \in (1, \bar{\lambda})$, the stability condition stated above is satisfied for the stationary solution associated with higher taxes and correspondingly lower inequality. The second equilibrium associated with higher inequality is unstable (see also Figures 7(a) and 7(b)). □

Proof of Proposition 5: The Jacobian matrix P of the two–dimensional system (14a) and (14b) is given by:

$$P = \begin{pmatrix} \beta^2(1-\tau_*)^2 - 2(1-\tau_*)(\sigma_u^2 + \beta^2\sigma_*^2)T_{\sigma^2} & -2(1-\tau_*)\left(\sigma_u^2 + \beta^2\sigma_*^2\right) T_\lambda \\ h & 0 \end{pmatrix}$$

The eigenvalues ν_1, ν_2 of P are given by the two roots of the characteristic equation:

$$0 = \nu^2 - \nu\left[\beta^2(1-\tau_*)^2 - 2(1-\tau_*)(\sigma_u^2 + \beta^2\sigma_*^2)T_{\sigma^2}\right] + 2(1-\tau_*)\left(\sigma_u^2 + \beta^2\sigma_*^2\right) T_\lambda h\,.$$

Let us define the function $f(\nu)$:

$$\begin{aligned} f(\nu) \equiv \nu^2 - \nu\left[\beta^2(1-\tau_*)^2 - 2(1-\tau_*)(\sigma_u^2 + \beta^2\sigma_*^2)T_{\sigma^2}\right] \\ = -2(1-\tau_*)\left(\sigma_u^2 + \beta^2\sigma_*^2\right) T_\lambda h \qquad \text{(A.6)} \end{aligned}$$

The function $f(\nu)$ on the left hand side of equation (A.6) is quadratic in ν with roots at $\nu = 0$ and at $\tilde{\nu} = \beta^2(1-\tau_*)^2 - 2(1-\tau_*)(\sigma_u^2 + \beta^2\sigma_*^2)T_{\sigma^2}$. Notice that, according to Proposition 4, (14a) displays stable dynamics for a given value of λ, which implies $|\tilde{\nu}| < 1$. Moreover, from Proposition 2 we know that $T_\lambda < 0$, such that the right hand side of equation (A.6) is always negative as long as $h < 0$. Depending on the value of h we are able to distinguish three cases for the roots associated with equation (A.6): (a) two real roots with modulus less than one, or (b) conjugate complex roots with modulus less than one, or (c) conjugate complex roots with modulus greater than one. For a Hopf–bifurcation to emerge at the bifurcation value $h = h^b$, the eigenvalues $\nu = \nu(h)$ of the Jacobian must satisfy the following conditions (cf. Azariadis, 1993, pp. 100):

(i) $|\nu(h^b)| = 1$

(ii) $\nu(h^b)^j \neq \pm 1$ for $j = 1, 2, 3, 4$

(iii) $\frac{d|\nu(h)|}{dh}\Big|_{h=h^b} \neq 0$

If these conditions are met, there is an invariant closed curve bifurcating from h^b.

In order to simplify the representation, let us write the above characteristic polynomial as $f(\nu) = \nu^2 - a\nu = hb$, where $0 < a < 1$ and $b > 0$ Regarding condition (i), the corresponding value of h^b such that $\nu(h^b) = 1$ is $h^b = -1/b$. In this case the roots are complex and can be written as $\nu_{1,2} = Re^{\pm i\theta}$, where $R = \sqrt{-h^b b} = 1$ and $\theta = a/2$. Since $|\nu(h^b)| = \sqrt{-hb}$, we have $\frac{d|\nu(h)|}{dh} = -b\,\frac{1}{2}\,(-hb)^{-1/2}$ and condition (iii) is satisfied too. With respect to condition (ii) which requires that the eigenvalues at h^b are not higher roots of unity, it is sufficient to mention, that this can always be ruled out by a slightly perturbation of the parameters of the model. □

Working Paper Series in Economics

(see www.uni-lueneburg.de/vwl/papers for a complete list)

No. 1: *Joachim Wagner:* Nascent and Infant Entrepreneurs in Germany. Evidence from the Regional Entrepreneurship Monitor (REM). March 2005

No. 2: *Ingrid Ott and Stephen Turnovsky:* Excludable and Non-Excludable Public Inputs: Consequences for Economic Growth, March 2005 (also published as CESifo Working Paper 1423)

No. 3: *Thomas Wein and Reimund Schwarze:* Is the Market Classification of Risk Always Efficient? - Evidence from German Third Party Motor Insurance, March 2005

No. 4: *Joachim Wagner:* Exports and Productivity: A Survey of the Evidence from Firm Level Data. March 2005

No. 5: *Christiane Clemens and Maik Heinemann:* Endogenous Redistributive Cycles — An Overlapping Generations Approach to Social Conflict and Cyclical Growth, March 2005

No. 6: *Christiane Clemens and Maik Heinemann:* On the Effects of Redistribution on Growth and Entrepreneurial Risk-Taking, March 2005

No. 7: *Thomas Wein:* Associations‘ Agreement and the Interest of the Network Suppliers — The Strategic Use of Structural Features, March 2005

No. 8: *Joachim Wagner:* Exports, Foreign Direct Investment, and Productivity: Evidence from German Firm Level Data. March 2005

No. 9: *Gabriel Desgranges and Maik Heinemann:* Strongly Rational Expectations Equilibria with Endogenous Acquisition of Information, March 2005

Universität Lüneburg
Institut für Volkswirtschaftslehre
Scharnhorststraße 1
D-21332 Lüneburg
Tel.: ++49 4131 78 2321
email: brodt@uni.lueneburg.de
www.uni-lueneburg.de/vwl/papers